

\mathcal{N} -fold Supersymmetric Quantum Mechanics with Reflections

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Abstract

We formulate \mathcal{N} -fold supersymmetry in quantum mechanical systems with reflection operators. As in the cases of other systems, they possess the two significant characters of \mathcal{N} -fold supersymmetry, namely, almost isospectrality and weak quasi-solvability. We construct explicitly the most general 1- and 2-fold supersymmetric quantum mechanical systems with reflections. In the case of $\mathcal{N} = 2$, we find that there are seven inequivalent such systems, three of which are characterized by three arbitrary functions having definite parity while the other four of which are by two. In addition, four of the seven inequivalent systems do not reduce to ordinary quantum systems without reflections. Furthermore, in certain particular cases, they are essentially equivalent to the most general two-by-two Hermitian matrix 2-fold supersymmetric quantum systems obtained previously by us.

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I. INTRODUCTION

Recently, supersymmetry (SUSY) was formulated for one-dimensional quantum mechanical systems with reflections in Ref. [1]. One of its characteristic features is that both a supersymmetric Hamiltonian and a supercharge component involve reflection operators. An intriguing aspect shown in this work is that exact eigenfunctions of such a system are expressed in terms of little -1 Jacobi polynomials which is one of a “missing” family of classical orthogonal polynomials [2]. Hence, it is interesting to study what kind of Hamiltonians involving reflection operators admit exact eigenfunctions which are expressible in terms of such a “missing” classical orthogonal polynomial system.

On the other hand, the framework of \mathcal{N} -fold SUSY [3–5] has been shown to be quite fruitful among several generalizations of ordinary SUSY especially since the establishment of its equivalence with weak quasi-solvability in Ref. [4], for a review see Ref. [6]. Until now, four different types have been established, namely, type A [7, 8], type B [9], type C [10], and type X_2 [11]. We note that almost all the models having essentially the same symmetry as \mathcal{N} -fold SUSY but called with other terminologies in the literature, such as Pöschl–Teller and Lamé potentials, are actually particular cases of type A \mathcal{N} -fold SUSY. To avoid confusion, we also note that \mathcal{N} -fold SUSY is different from *nonlinear SUSY* which has been long employed since the work by Samuel and Wess [12] in 1983 to indicate nonlinearly realized SUSY originated from the work by Akulov and Volkov [13] in 1972. For recent works on nonlinear SUSY, see, e.g., Ref. [14] and references cited therein. Due to the facts that the $\mathcal{N} = 1$ case corresponds to ordinary SUSY and that exact solvability always means weak quasi-solvability, the framework of \mathcal{N} -fold SUSY enables us to formulate systematically ordinary SUSY and exactly solvable quantum systems as its particular cases. In fact, we successfully formulated in Ref. [15] \mathcal{N} -fold SUSY in quantum mechanical matrix models as a generalization of ordinary SUSY quantum mechanical matrix models in, e.g., Ref. [16] and references cited therein. Hence, it is quite natural to ask whether a formulation of \mathcal{N} -fold SUSY is possible for quantum mechanical systems with reflection operators. To the best of our knowledge, there have been no such attempts in the existing literature.

In this article, we formulate for the first time \mathcal{N} -fold SUSY for quantum mechanical systems with reflection operators for all positive integral \mathcal{N} . To see concretely what kinds of forms such systems must have, we construct the most general 1- and 2-fold SUSY systems by solving directly all the conditions for the respective SUSYs. In the case of $\mathcal{N} = 1$, we find in particular that ordinary SUSY algebra can be realized under a less restrictive condition than the one presupposed in Ref. [1]. In the case of $\mathcal{N} = 2$, we find that there are seven inequivalent systems, three of which are characterized by three arbitrary functions having definite parity while the other four of which are by two. In addition, we also find that four of the seven inequivalent systems do not admit a reduction to 2-fold SUSY ordinary quantum systems without reflection operators. Furthermore, in certain particular cases, they are essentially equivalent to the most general two-by-two Hermitian matrix 2-fold supersymmetric quantum systems obtained previously by us in Ref. [15].

We organize this article as follows. In the next section, we first summarize fundamental formulas which are frequently needed for calculations involving both differential and reflection operators. Then, we generically define \mathcal{N} -fold SUSY in quantum mechanical systems with reflections. In Section III, we present the most general results in the $\mathcal{N} = 1$ case which corresponds to ordinary SUSY. We also clarify the relation between our formalism and the SUSY QM with reflections formulated in Ref. [1]. In Section IV, we investigate in detail

the $\mathcal{N} = 2$ case. We explicitly solve all the conditions for 2-fold SUSY to obtain the most general form of the latter systems. In the last section, we refer to several future issues to be followed after this work.

II. PRELIMINARIES AND GENERAL SETTING

First of all, let \mathcal{P} denote a reflection or parity operator whose action on an element of a linear function space \mathfrak{F} is defined by

$$\mathcal{P} \cdot \psi(q) = \psi(-q) := \psi_{\mathcal{P}}(q). \quad (1)$$

We will hereafter use the last notation $\psi_{\mathcal{P}}$ frequently especially when we will omit the argument of the function under consideration. On the other hand, if $f(q)$ is a multiplicative operator in \mathfrak{F} , we have instead an operator relation as

$$\mathcal{P}f(q) = f_{\mathcal{P}}(q)\mathcal{P}. \quad (2)$$

Another important operator relation is the one between reflection and differential operators

$$\frac{d}{dq}\mathcal{P} = -\mathcal{P}\frac{d}{dq}. \quad (3)$$

Due to the latter anti-commutativity, we have in particular

$$(\psi_{\mathcal{P}})'(q) = \frac{d\psi_{\mathcal{P}}(q)}{dq} = -\mathcal{P} \cdot \frac{d\psi(q)}{dq} = -(\psi')_{\mathcal{P}}(q). \quad (4)$$

Any function $f(q)$ admits the decomposition into its even and odd parts, denoted respectively by $f_+(q)$ and $f_-(q)$, as

$$f(q) = f_+(q) + f_-(q), \quad 2f_{\pm}(q) = f(q) \pm f_{\mathcal{P}}(q). \quad (5)$$

It is evident from (4) and (5) that

$$(f')_{\pm}(q) = (f_{\mp})'(q). \quad (6)$$

The formulas (1)–(6) are fundamental tools for dealing with differential operators with reflections.

A quantum mechanical system we shall consider here is a pair of Schrödinger operators which involve reflection operators as follows:

$$H^{\pm} = -\frac{1}{2}\frac{d^2}{dq^2} + V_0^{\pm}(q) + V_1^{\pm}(q)\mathcal{P}, \quad (7)$$

where the potential functions $V_0^{\pm}(q)$ and $V_1^{\pm}(q)$ are to be determined later. Let us introduce a pair of linear differential operators of order \mathcal{N} with reflection operators

$$P_{\mathcal{N}}^{-} = \frac{d^{\mathcal{N}}}{dq^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} \left[w_k^{[\mathcal{N}]}(q) + v_k^{[\mathcal{N}]}(q)\mathcal{P} \right] \frac{d^k}{dq^k}, \quad (8a)$$

$$P_{\mathcal{N}}^{+} = (P_{\mathcal{N}}^{-})^{\text{T}} = (-1)^{\mathcal{N}} \frac{d^{\mathcal{N}}}{dq^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} (-1)^k \frac{d^k}{dq^k} \left[w_k^{[\mathcal{N}]}(q) + \mathcal{P}v_k^{[\mathcal{N}]}(q) \right], \quad (8b)$$

where $w_k^{[\mathcal{N}]}(q)$ and $v_k^{[\mathcal{N}]}(q)$ ($k = 0, \dots, \mathcal{N} - 1$) are in general complex analytic functions, and the superscript T denotes transposition. We will hereafter omit the superscript $[\mathcal{N}]$ for the simplicity unless the omission may cause confusion or ambiguity. Then, the system (7) is said to be \mathcal{N} -fold supersymmetric with respect to (8) if the following relations are all satisfied:

$$P_{\mathcal{N}}^{\mp} H^{\mp} - H^{\pm} P_{\mathcal{N}}^{\mp} = 0, \quad (9)$$

$$P_{\mathcal{N}}^{\mp} P_{\mathcal{N}}^{\pm} = 2^{\mathcal{N}} \left[(H^{\pm} + C_0)^{\mathcal{N}} + \sum_{k=1}^{\mathcal{N}-1} C_k (H^{\pm} + C_0)^{\mathcal{N}-k-1} \right], \quad (10)$$

where C_k ($k = 0, \dots, \mathcal{N} - 1$) are constant multiplicative operators with reflections

$$C_k = C_{k0} + C_{k1} \mathcal{P}. \quad (11)$$

The two intertwining relations in (9) are related by transposition if both the potential terms $V_1^+(q)$ and $V_1^-(q)$ are even and thus commute with \mathcal{P} , since in the latter case both the Hamiltonians H^+ and H^- are invariant under transposition, $(H^{\pm})^T = H^{\pm}$. As is usual, we can express an \mathcal{N} -fold SUSY system in a unified way by introducing a superHamiltonian \mathbf{H} and a pair of \mathcal{N} -fold supercharges $\mathbf{Q}_{\mathcal{N}}^{\pm}$ as

$$\mathbf{H} = H^- \psi^- \psi^+ + H^+ \psi^+ \psi^-, \quad \mathbf{Q}_{\mathcal{N}}^{\pm} = P_{\mathcal{N}}^{\mp} \psi^{\pm}, \quad (12)$$

where ψ^{\pm} are fermionic variables satisfying $(\psi^{\pm})^2 = 0$ and $\{\psi^+, \psi^-\} = 1$. Then, the \mathcal{N} -fold SUSY relations (9) and (10) are summarized in \mathcal{N} -fold superalgebra

$$[\mathbf{Q}_{\mathcal{N}}^{\pm}, \mathbf{H}] = 0, \quad \{\mathbf{Q}_{\mathcal{N}}^+, \mathbf{Q}_{\mathcal{N}}^-\} = 2^{\mathcal{N}} \left[(\mathbf{H} + C_0)^{\mathcal{N}} + \sum_{k=1}^{\mathcal{N}-1} C_k (\mathbf{H} + C_0)^{\mathcal{N}-k-1} \right]. \quad (13)$$

We note that in the case of $\mathcal{N} = 1$ the above definition of \mathcal{N} -fold SUSY is slightly different from the SUSY QM with reflections in Ref. [1]. The exact relation between our P_1^- and a supercharge component Q in the latter reference, Eq. (2.11), is $P_1^- = \sqrt{2} Q \mathcal{P}$ with $w_0 = U$, $v_0 = V$, and $\mathcal{P} = R$. In particular, our \mathcal{N} -fold supercharge components $P_{\mathcal{N}}^{\pm}$ do not possess formal Hermiticity in contrast with Q in the latter. The relations between our 1-fold SUSY pair of Hamiltonians and a SUSY Hamiltonian H in the latter are $H^+ = H$ and $H^- = \mathcal{P} H \mathcal{P}$. In particular, $H^- = H^+$ if H^+ commutes with a reflection operator \mathcal{P} .

It is evident from the definition that \mathcal{N} -fold SUSY quantum systems with reflections (7)–(11) reduce to ones without reflections if $V_1^{\pm}(q) = v_k(q) = C_{01} = 0$ for all $k = 0, \dots, \mathcal{N} - 1$. As in the case without reflections, the first relation (9) immediately implies almost isospectrality of H^{\pm} and *weak quasi-solvability* $H^{\pm} \ker P_{\mathcal{N}}^{\pm} \subset \ker P_{\mathcal{N}}^{\pm}$.

III. ORDINARY SUSY

In this section, we shall examine the $\mathcal{N} = 1$ case, namely, ordinary SUSY QM with reflections. Components of supercharges are given by

$$P_1^- = \frac{d}{dq} + w_0(q) + v_0(q) \mathcal{P}, \quad P_1^+ = -\frac{d}{dq} + w_0(q) + \mathcal{P} v_0(q). \quad (14)$$

A direct calculation immediately yields

$$P_1^- P_1^+ = -\frac{d^2}{dq^2} - 2v_{0+} \mathcal{P} \frac{d}{dq} + w'_0 + (w_0)^2 + (v_0)^2 + (-(v'_0)_{\mathcal{P}} + w_0 v_{0\mathcal{P}} + w_{0\mathcal{P}} v_0) \mathcal{P}, \quad (15)$$

$$P_1^+ P_1^- = -\frac{d^2}{dq^2} + 2v_{0+} \mathcal{P} \frac{d}{dq} - w'_0 + (w_0)^2 + (v_{0\mathcal{P}})^2 + (-v'_0 + w_0 v_0 + w_{0\mathcal{P}} v_{0\mathcal{P}}) \mathcal{P}. \quad (16)$$

Hence, they are of the form (7) if and only if

$$2v_{0+}(q) = v_0(q) + v_{0\mathcal{P}}(q) = 0, \quad (17)$$

that is, $v_0(q)$ is an odd function $v_0(q) = v_{0-}(q)$. Under the latter condition, the \mathcal{N} -fold superalgebra (10) in the case of $\mathcal{N} = 1$ holds and the potential terms in (7) are expressed as

$$2V_0^\pm = \pm w'_0 + (w_0)^2 + (v_{0-})^2 - 2C_{00}, \quad (18)$$

$$2V_1^\pm = -(v_{0-})' \mp 2w_{0-} v_{0-} - 2C_{01}, \quad (19)$$

where C_{00} and C_{01} are constants defined by (11). The intertwining relation (9) is trivially satisfied. We note that $V_1^\pm(q)$ is automatically even for an arbitrary $w_0(q)$. In this respect, it is also worth mentioning that the evenness of $w_0(q)$ is not inevitable for SUSY although it was presupposed in Ref. [1]. When $w_0(q)$ is even,

IV. 2-FOLD SUSY

Next, we shall proceed to the $\mathcal{N} = 2$ case where components of 2-fold supercharges are given by

$$P_2^- = \frac{d^2}{dq^2} + [w_1(q) + v_1(q)\mathcal{P}] \frac{d}{dq} + w_0(q) + v_0(q)\mathcal{P}, \quad (20a)$$

$$P_2^+ = \frac{d^2}{dq^2} - \frac{d}{dq} [w_1(q) + \mathcal{P}v_1(q)] + w_0(q) + \mathcal{P}v_0(q). \quad (20b)$$

Before investigating the intertwining relation (9) for $\mathcal{N} = 2$, we first note that a direct calculation shows (see Eqs. (A1) and (A2) in Appendix for the full formulas)

$$P_2^\mp P_2^\pm = \partial^4 + 2v_{1+} \mathcal{P} \partial^3 + O(\partial^2), \quad (21)$$

where $O(\partial^2)$ denotes a linear differential operator of at most second order. Hence, it is necessary that the function $v_1(q)$ is odd

$$2v_{1+}(q) = v_1(q) + v_{1\mathcal{P}}(q) = 0, \quad (22)$$

for satisfying the 2-fold superalgebra (10), that is, $v_1(q) = v_{1-}(q)$. Under the latter condition, the second-order intertwining relation $P_2^- H^- - H^+ P_2^- = 0$ holds if and only if the following set of conditions are satisfied:

$$V_0^+ - V_0^- = w'_1, \quad (23)$$

$$V_1^+ - V_1^- = -(v_{1-})', \quad (24)$$

$$w_1'' + 2w'_0 + 4V_0^{-'} - 2w_1(V_0^+ - V_0^-) + 2v_{1-}(V_1^+ - (V_1^-)_{\mathcal{P}}) = 0, \quad (25)$$

$$(v_{1-})'' - 2v'_0 - 4V_1^{-'} - 2v_{1-}(V_0^+ - (V_0^-)_{\mathcal{P}}) - 2w_{1\mathcal{P}}V_1^+ - 2w_1V_1^- = 0, \quad (26)$$

$$w_0'' + 2V_0^{-''} + 2w_1V_0^{-'} - 2v_{1-}((V_1^-)_{\mathcal{P}})' - 2v_{0\mathcal{P}}V_1^+ + 2v_0(V_1^-)_{\mathcal{P}} - 2w_0(V_0^+ - V_0^-) = 0, \quad (27)$$

$$v_0'' + 2V_1^{-''} + 2w_1V_1^{-'} - 2v_{1-}((V_0^-)_{\mathcal{P}})' - 2w_{0\mathcal{P}}V_1^+ + 2w_0V_1^- - 2v_0(V_0^+ - (V_0^-)_{\mathcal{P}}) = 0. \quad (28)$$

On the other hand, using the formula

$$4(H^\pm)^2 = \frac{d^4}{dq^4} - 4(V_0^\pm + V_1^\pm \mathcal{P}) \frac{d^2}{dq^2} - 4(V_0^{\pm'} - V_1^{\pm'} \mathcal{P}) \frac{d}{dq} - 2[V_0^{\pm''} - 2(V_0^\pm)^2 - 2V_1^\pm(V_1^\pm)_\mathcal{P}] - 2[V_1^{\pm''} - 4(V_0^\pm)_+ V_1^\pm] \mathcal{P}, \quad (29)$$

we find that the 2-fold superalgebra $P_2^\mp P_2^\pm = 4[(H^\pm + C_0)^2 + C_1]$ holds for the upper sign if and only if

$$4V_0^+ = 3w_1' - 2w_0 + (w_1)^2 + (v_{1-})^2 - 4C_{00}, \quad (30)$$

$$4V_1^+ = -3(v_{1-})' - 2v_{0+} - 2w_{1-}v_{1-} - 4C_{01}, \quad (31)$$

$$4V_0^{+'} = 3w_1'' - 2w_0' + 2w_1w_1' + 2v_{1-}(v_{1-})', \quad (32)$$

$$4V_1^{+'} = -3(v_{1-})'' - 2(v_{0\mathcal{P}})' + 2(w_{1\mathcal{P}})'v_{1-} - 2w_1(v_{1-})' - 2w_{0-}v_{1-} - w_1v_{0\mathcal{P}} - w_{1\mathcal{P}}v_0, \quad (33)$$

$$2V_0^{+''} - 4(V_0^+ + C_{00})^2 - 4(V_1^+ + C_{01})((V_1^+)_\mathcal{P} + C_{01}) - 4C_{10} = w_1''' - w_0'' + w_1w_1'' + v_{1-}(v_{1-})'' + w_1'w_0 - w_1w_0' - (v_{1-})'v_0 + v_{1-}v_0' - (w_0)^2 - (v_0)^2, \quad (34)$$

$$2V_1^{+''} - 8((V_0^+)_+ + C_{00})(V_1^+ + C_{01}) - 4C_{11} = -(v_{1-})''' - (v_{0\mathcal{P}})'' + (w_{1\mathcal{P}})''v_{1-} - w_1(v_{1-})'' - (w_{1\mathcal{P}})'v_0 - w_1(v_{0\mathcal{P}})' + (w_{0\mathcal{P}})'v_{1-} - w_0(v_{1-})' - w_0v_{0\mathcal{P}} - w_{0\mathcal{P}}v_0, \quad (35)$$

and for the lower sign if and only if

$$4V_0^- = -w_1' - 2w_0 + (w_1)^2 + (v_{1-})^2 - 4C_{00}, \quad (36)$$

$$4V_1^- = (v_{1-})' - 2v_{0+} - 2w_{1-}v_{1-} - 4C_{01}, \quad (37)$$

$$4V_0^{-'} = -w_1'' - 2w_0' + 2w_1w_1' + 2v_{1-}(v_{1-})', \quad (38)$$

$$4V_1^{-'} = (v_{1-})'' - 2v_0' - 2(w_{1-})'v_{1-} - 2w_{1-}(v_{1-})' + w_1v_0 + w_{1\mathcal{P}}v_{0\mathcal{P}} + 2w_{0-}v_{1-}, \quad (39)$$

$$2V_0^{-''} - 4(V_0^- + C_{00})^2 - 4(V_1^- + C_{01})((V_1^-)_\mathcal{P} + C_{01}) - 4C_{10} = -w_0'' + w_1'w_0 + w_1w_0' - (v_{1-})'v_{0\mathcal{P}} - v_{1-}(v_{0\mathcal{P}})' - (w_0)^2 - (v_{0\mathcal{P}})^2, \quad (40)$$

$$2V_1^{-''} - 8((V_0^-)_+ + C_{00})(V_1^- + C_{01}) - 4C_{11} = -v_0'' + w_1'v_0 + w_1v_0' - (w_{0\mathcal{P}})'v_{1-} - w_{0\mathcal{P}}(v_{1-})' - w_0v_0 - w_{0\mathcal{P}}v_{0\mathcal{P}}. \quad (41)$$

The formulas (30), (31), (36), and (37) determine the form of all the potential terms V_0^\pm and V_1^\pm . In addition, they are automatically compatible with (23)–(25), (32), and (38). From (31) and (37), we see that both the potential terms $V_1^+(q)$ and $V_1^-(q)$ are even and thus we do not need to check the other intertwining relation $P_2^+ H^+ - H^- P_2^+ = 0$ in (9). Hence, there remain nine conditions, (26)–(28), (33)–(35), and (39)–(41) to be investigated. Let us first begin with (26), (33), and (39). By the substitution of (30), (31), (36), and (37) into them, they read as

$$2(v_{0-})' + (w_{1+})'v_{1-} - w_{1+}(v_{1-})' - 2w_{1+}v_{0+} - 2w_{0-}v_{1-} - 4C_{01}w_{1+} = 0, \quad (42)$$

$$(v_{0-})' + (w_{1+})'v_{1-} - w_{1+}(v_{1-})' - w_{1+}v_{0+} + w_{1-}v_{0-} - w_{0-}v_{1-} = 0, \quad (43)$$

$$(v_{0-})' - w_{1+}v_{0+} - w_{1-}v_{0-} - w_{0-}v_{1-} = 0. \quad (44)$$

It is easy to check that they are compatible with each other if and only if

$$C_{01}w_{1+} = 0. \quad (45)$$

From (43) and (44), we obtain

$$(w_{1+})'v_{1-} - w_{1+}(v_{1-})' + 2w_{1-}v_{0-} = 0. \quad (46)$$

Hence, the conditions (26), (33), and (39) are equivalent to (44)–(46). We note in particular that Eq. (46) enables us to express w_{1-} (or v_{0-}) in terms of w_{1+} , v_{1-} , and v_{0-} (or w_{1-}), respectively. Next, we shall investigate (27), (34), and (40). Substituting (30), (31), (36), and (37) into them, we have the three conditions (A3)–(A5) presented in Appendix. We can easily check that they are equivalent to the following set of conditions:

$$\begin{aligned} & 2w_1w_1'' - (w_1')^2 + 2v_{1-}(v_{1-})'' - ((v_{1-})')^2 - 4v_{1-}(v_{0-})' + 4(v_{0-})^2 - 2(w_1)^2w_1' \\ & + 4(w_1)^2w_0 - 2w_1'(v_{1-})^2 - 4w_{1-}v_{1-}(v_{1-})' - 8w_{1-}v_{1-}v_{0+} + 4w_0(v_{1-})^2 \\ & - (w_1)^4 - 2[(w_1)^2 + 2(w_{1-})^2](v_{1-})^2 - (v_{1-})^4 - 16C_{10} = 0, \end{aligned} \quad (47)$$

$$\begin{aligned} & w_1''' - w_1w_1'' - 2(w_1')^2 + 4w_1'w_0 + 2w_1w_0' - v_{1-}(v_{1-})'' - 2((v_{1-})')^2 \\ & - 2(v_{1-})'(2v_{0+} - v_{0-}) - 2v_{1-}(v_{0+})' + 4v_{0+}v_{0-} - 2(w_1)^2w_1' \\ & - 2w_1'(v_{1-})^2 - 4w_{1-}v_{1-}(v_{1-})' = 0, \end{aligned} \quad (48)$$

$$C_{01}v_{0-} = 0, \quad (49)$$

where Eq. (46) has been applied for the derivation of the last formula. Now, the remaining conditions to be examined are (28), (35), and (41). Substituting (30), (31), (36), and (37) into them, we have the three conditions (A6)–(A8) presented in Appendix. We can easily check that they are equivalent to the following set of conditions:

$$\begin{aligned} & w_1''v_{1-} - (w_{1-})'(v_{1-})' - w_{1\mathcal{P}}(v_{1-})'' + 2(w_{1-})'v_{0-} + 2w_1(v_{0-})' - 2w_{1-}(w_{1-})'v_{1-} \\ & - [(w_{1+})^2 + (w_{1-})^2](v_{1-})' - 2[(w_{1+})^2 + (w_{1-})^2]v_{0+} + 4w_{1-}w_{0+}v_{1-} - (v_{1-})^2(v_{1-})' \\ & - 2(v_{1-})^2v_{0+} - 2[(w_{1+})^2 + (w_{1-})^2]w_{1-}v_{1-} - 2w_{1-}(v_{1-})^3 + 8C_{11} = 0, \end{aligned} \quad (50)$$

$$\begin{aligned} & (v_{1-})''' + 2(v_{0-})'' + (w_{1\mathcal{P}})''v_{1-} - 4(w_{1-})'(v_{1-})' - w_1(v_{1-})'' - 2(w_{1+})'v_0 \\ & - 4(w_{1-})'v_{0+} - 2w_1(v_{0+})' + 2(w_{0\mathcal{P}})'v_{1-} + 2(2w_{0+} - w_{0-})(v_{1-})' + 4w_{0-}v_{0-} \\ & - 4w_{1-}(w_{1-})'v_{1-} - 2[(w_{1+})^2 + (w_{1-})^2](v_{1-})' - 2(v_{1-})^2(v_{1-})' = 0, \end{aligned} \quad (51)$$

$$C_{01}w_{0-} = 0, \quad (52)$$

where Eqs. (44) and (46) have been applied for the derivation of the last formula. From (45), (49), and (52), we conclude that

$$C_{01} = 0 \quad \text{or} \quad w_{1+} = v_{0-} = w_{0-} = 0. \quad (53)$$

To analyze (47), (48), (50), and (51), we first note that the equalities must hold for their even and odd parts separately since we obtain another set of equalities by applying a reflection operator to them. The even and odd parts of them are explicitly presented in Appendix, (A9)–(A16). It is apparent that (A14) and (A16) are automatically satisfied under the conditions (44) and (46). Hence, the remaining problem is to solve (A9)–(A13) and (A15) simultaneously. However, they are not independent under the conditions (44) and (46). In fact, we can check that the following combinations

$$\begin{aligned} & 2w_{1-} \times (\text{A11}) + 2w_{1+} \times (\text{A12}) + 2v_1 \times (\text{A15}) + 4v_0 \times [2 \times (\text{A4}) + (\text{A6})], \\ & w_{1+} \times (\text{A11}) + w_{1-} \times (\text{A12}) - (v_1' + 2v_{0+}) \times (\text{A6}), \\ & v_1 \times (\text{A11}) + w_{1-} \times (\text{A15}) + [(w_{1+})' - 2w_{0-}] \times (\text{A6}), \end{aligned}$$

are identical with the equations obtained by differentiating (A9), (A10), and (A13), respectively. In other words, the set of equations (A9), (A10), and (A13) are equivalent to the set of equations (A11), (A12), and (A15) under the conditions (44) and (46). Therefore, the remaining task we should settle is now to solve only the former with (44), (46), and (53). In what follows, we shall analyze separately the two cases of $C_{01} = 0$ and $C_{01} \neq 0$.

A. The $C_{01} = 0$ Case

In this case, the condition (53) does not provide any constraint on the three functions w_{1+} , v_{0-} , and w_{0-} . Hence, all that we should do is to solve (A9), (A10), and (A13) under the two conditions (44) and (46). It is actually easy since they can be regarded as simultaneous linear equations for w_{0+} , w_{0-} , and v_{0+} :

$$2A \begin{pmatrix} w_{0+} \\ w_{0-} \\ v_{0+} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad (54)$$

where f_i ($i = 1, 2, 3$) which only depend on w_{1+} , w_{1-} , and v_{1-} are explicitly presented in (A17)–(A19) while the 3×3 matrix A is given by

$$A = \begin{pmatrix} 2[(w_{1+})^2 + (w_{1-})^2 + (v_{1-})^2] & 4w_{1+}w_{1-} & -4w_{1-}v_{1-} \\ 2w_{1+}w_{1-} & (w_{1+})^2 + (w_{1-})^2 & -w_{1+}v_{1-} \\ 2w_{1-}v_{1-} & w_{1+}v_{1-} & -(w_{1-})^2 - (v_{1-})^2 \end{pmatrix}. \quad (55)$$

Hence, we must treat the problem separately according to the value of the determinant of A :

$$\det A = -2(w_{1-})^2 [(w_{1+})^2 - (w_{1-})^2 + (v_{1-})^2]^2. \quad (56)$$

Case 1. $(w_{1-})^2 \neq (w_{1+})^2 + (v_{1-})^2$ and $w_{1-} \neq 0$:

In the non-degenerate case $\det A \neq 0$, they are uniquely solved as

$$\begin{aligned} -2(\det A) w_{0+} &= (w_{1-})^2 [(w_{1+})^2 + (w_{1-})^2 + (v_{1-})^2] f_1 - 4w_{1+}(w_{1-})^3 f_2 \\ &\quad - 4(w_{1-})^3 v_{1-} f_3, \end{aligned} \quad (57)$$

$$\begin{aligned} -(\det A) w_{0-} &= -w_{1+}(w_{1-})^3 f_1 + [(w_{1+})^2(w_{1-})^2 + (w_{1-})^4 + (w_{1+})^2(v_{1-})^2 \\ &\quad - 2(w_{1-})^2(v_{1-})^2 + (v_{1-})^4] f_2 - [(w_{1+})^2 - 3(w_{1-})^2 + (v_{1-})^2] w_{1+}v_{1-} f_3, \end{aligned} \quad (58)$$

$$\begin{aligned} -(\det A) v_{0+} &= (w_{1-})^3 v_{1-} f_1 + [(w_{1+})^2 - 3(w_{1-})^2 + (v_{1-})^2] w_{1+}v_{1-} f_2 \\ &\quad - [(w_{1+})^4 - 2(w_{1+})^2(w_{1-})^2 + (w_{1-})^4 + (w_{1+})^2(v_{1-})^2 + (w_{1-})^2(v_{1-})^2] f_3, \end{aligned} \quad (59)$$

Therefore, the most general 2-fold SUSY quantum systems with reflections composed of H^\pm in (7) and P_2^\pm in (20) are entirely expressible solely in terms of three arbitrary functions having definite parity w_{1+} , w_{1-} , and v_{1-} by using (46) and (57)–(59) in the non-degenerate case. When $v_{1-} = v_{0+} = 0$, then $v_{0-} = 0$ from (46) and the set of identities (57)–(59) reduces to

$$4(w_1)^2 w_0 = -2w_1(w_1)'' + ((w_1)')^2 + 2(w_1)^2(w_1)' + (w_1)^4 + 16C_{10}, \quad C_{11} = 0. \quad (60)$$

Hence, the systems in this case reduces to the most general 2-fold SUSY ordinary quantum systems without reflections in Refs. [4, 17, 18].

Case 2. $(w_{1-})^2 = (w_{1+})^2 + (v_{1-})^2 \neq 0$:

Next, we shall examine the degenerate case

$$(w_{1-})^2 = (w_{1+})^2 + (v_{1-})^2 \neq 0. \quad (61)$$

In this case, the three equations (A9), (A10), and (A13) are not linearly independent and are equivalent to the two equations

$$8(w_{1-})^2(w_{1+}w_{0+} + w_{1-}w_{0-}) = -w_{1+}f_1 + 4w_{1-}f_2, \quad (62a)$$

$$8(w_{1-})^2(v_{1-}w_{0+} - w_{1-}v_{0+}) = -v_{1-}f_1 + 4w_{1-}f_3, \quad (62b)$$

with the constraint

$$w_{1-}f_1 - 2w_{1+}f_2 - 2v_{1-}f_3 = 16(C_{10}w_{1-} + C_{11}v_{1-}) = 0. \quad (63)$$

We note that we have the following interesting formula by using (46) and (61):

$$4(v_{0-})^2 = ((w_{1+})')^2 - ((w_{1-})')^2 + ((v_{1-})')^2. \quad (64)$$

We first show that $C_{10} = C_{11} = 0$ to satisfy the constraint (63). Suppose $C_{11} \neq 0$ since $C_{11} = 0$ inevitably means $C_{10} = 0$ due to the assumption $w_{1-} \neq 0$ in this case. Then, we have $v_{1-} = -C_{10}w_{1-}/C_{11} := \tilde{C}_1 w_{1-}$ from (63). Substituting it into the current assumption $(w_{1-})^2 = (w_{1+})^2 + (v_{1-})^2$, we obtain $w_{1+} = \tilde{C}_2 w_{1-}$ with $(\tilde{C}_1)^2 + (\tilde{C}_2)^2 = 1$. But w_{1+} and w_{1-} are even and odd analytic functions, respectively, and thus we must conclude that $w_{1+} = w_{1-} = 0$, which contradicts with the assumption $w_{1-} \neq 0$. Hence, we eventually have $C_{10} = C_{11} = 0$.

Finally, using (61), (62), and (64), we can eliminate four functions, e.g., w_{1-} , w_{0-} , v_{0+} , and v_{0-} , to express the most general 2-fold SUSY quantum systems with reflections in this case in terms of the remaining three arbitrary functions having definite parity, e.g., w_{1+} , v_{1-} , and w_{0+} .

We note that the systems in this case do not admit a reduction to ordinary quantum systems without reflections. Indeed, if we put $v_{1-} = 0$, then $v_{0-} = 0$ from (46) since $w_{1-} \neq 0$ by the assumption, and thus $(w_{1+})' = \pm(w_{1-})'$ from (64). But $(w_{1+})'$ and $(w_{1-})'$ are odd and even analytic functions, and thus it is inevitable that $(w_{1+})' = (w_{1-})' = 0$. As any non-zero constant is an even function, we must conclude that $w_{1-} = 0$, which contradicts the assumption $w_{1-} \neq 0$. Hence, the systems in this case have no reductions to ordinary quantum systems without reflections.

Case 3: $w_{1-} = 0$:

Next, we shall examine the other degenerate case $w_{1-} = 0$. In this case, the potential terms V_0^\pm , V_1^\pm and the 2-fold supercharge component P_2^- read from (30), (31), (36), (37),

and (20) as

$$4V_0^+ = 3(w_{1+})' - 2w_{0+} - 2w_{0-} + (w_{1+})^2 + (v_{1-})^2 - 4C_{00}, \quad (65a)$$

$$4V_1^+ = -3(v_{1-})' - 2v_{0+}, \quad (65b)$$

$$4V_0^- = -(w_{1+})' - 2w_{0+} - 2w_{0-} + (w_{1+})^2 + (v_{1-})^2 - 4C_{00}, \quad (65c)$$

$$4V_1^- = (v_{1-})' - 2v_{0+}, \quad (65d)$$

$$P_2^- = \frac{d^2}{dq^2} + (w_{1+} + v_{1-}\mathcal{P})\frac{d}{dq} + w_{0+} + w_{0-} + (v_{0+} + v_{0-})\mathcal{P}. \quad (65e)$$

The conditions (44), (46), (A9), (A10), and (A13), which are all that we must manage, read as

$$(v_{0-})' - w_{1+}v_{0+} - w_{0-}v_{1-} = 0, \quad (66a)$$

$$(w_{1+})'v_{1-} - w_{1+}(v_{1-})' = 0, \quad (66b)$$

$$4[(w_{1+})^2 + (v_{1-})^2]w_{0+} = -2w_{1+}(w_{1+})'' + ((w_{1+})')^2 - 2v_{1-}(v_{1-})'' + ((v_{1-})')^2 - 4(v_{0-})^2 + (w_{1+})^4 + 2(w_{1+})^2(v_{1-})^2 + (v_{1-})^4 + 16C_{10}, \quad (66c)$$

$$2(w_{1+})^2w_{0-} - 2w_{1+}v_{1-}v_{0+} = (w_{1+})^2(w_{1+})' + w_{1+}v_{1-}(v_{1-})', \quad (66d)$$

$$2w_{1+}v_{1-}w_{0-} - 2(v_{1-})^2v_{0+} = w_{1+}(w_{1+})'v_{1-} + (v_{1-})^2(v_{1-})' - 8C_{11}. \quad (66e)$$

The second equality means that the two functions w_{1+} and v_{1-} are linearly dependent unless at least either of them vanishes. But they cannot be linearly dependent since w_{1+} and v_{1-} are even and odd analytic functions, respectively. Hence, we conclude that $w_{1+}v_{1-} = 0$. It turns out that we have three inequivalent solutions to them as the followings.

Case 3-1. $w_{1+} \neq 0$ and $v_{1-} = 0$:

In this case, the set of the conditions (66) are solved as

$$\begin{aligned} w_{0+} &= -\frac{(w_{1+})''}{2w_{1+}} + \frac{((w_{1+})')^2}{4(w_{1+})^2} - \frac{(v_{0-})^2}{(w_{1+})^2} + \frac{(w_{1+})^2}{4} + \frac{4C_{10}}{(w_{1+})^2}, \\ w_{0-} &= \frac{(w_{1+})'}{2}, \quad v_{0+} = \frac{(v_{0-})'}{w_{1+}}, \quad C_{11} = 0. \end{aligned} \quad (67)$$

Hence, the most general 2-fold SUSY quantum systems with reflections in this case are expressed in terms of the two arbitrary functions having definite parity w_{1+} and v_{0-} by the substitution of (67) into (65). When $v_{0-} = 0$, it is inevitable that $v_{0+} = 0$ from (67). Then, we have

$$w_0 = \frac{(w_{1+})^2}{4} - \frac{(w_{1+})''}{2w_{1+}} + \frac{((w_{1+})')^2}{4(w_{1+})^2} + \frac{(w_{1+})^2}{4} + \frac{4C_{10}}{(w_{1+})^2} + \frac{(w_{1+})'}{2}, \quad (68)$$

and the systems (65) with (68) exactly reduce to the general 2-fold SUSY ordinary quantum systems without reflections in Refs. [4, 17, 18] but with the restriction $w_1 = w_{1+}$.

Case 3-2. $w_{1+} = 0$ and $v_{1-} \neq 0$:

In this case, the set of the conditions (66) are solved as

$$\begin{aligned} w_{0+} &= -\frac{(v_{1-})''}{2v_{1-}} + \frac{((v_{1-})')^2}{4(v_{1-})^2} - \frac{(v_{0-})^2}{(v_{1-})^2} + \frac{(v_{1-})^2}{4} + \frac{4C_{10}}{(v_{1-})^2}, \\ w_{0-} &= \frac{(v_{0-})'}{v_{1-}}, \quad v_{0+} = -\frac{(v_{1-})'}{2} + \frac{4C_{11}}{(v_{1-})^2}. \end{aligned} \quad (69)$$

Hence, the most general 2-fold SUSY quantum systems with reflections in this case are expressed in terms of the two arbitrary functions having definite parity v_{1-} and v_{0-} by the substitution of (69) into (65). In contrast to the previous Case 3-1, the assumption $v_{1-} \neq 0$ does not admit of a reduction of the systems to ordinary quantum systems without reflections.

Case 3-3. $w_{1+} = v_{1-} = 0$:

In this case, the set of the conditions (66) are solved as

$$(v_{0-})^2 = 4C_{10}, \quad C_{11} = 0. \quad (70)$$

Hence, the most general 2-fold SUSY quantum systems with reflections in this case are expressed in terms of the three arbitrary functions having definite parity w_{0+} , w_{0-} , and v_{0+} by the substitution of (70) into (65). This case is almost trivial since we have from (65)

$$\begin{aligned} 2V_0^+ &= 2V_0^- = -w_0 - 2C_{00}, \quad 2V_1^+ = 2V_1^- = -v_{0+}, \\ P_2^- &= \frac{d^2}{dq^2} + w_0 + \left(v_{0+} \pm 2\sqrt{C_{10}}\right) \mathcal{P}, \end{aligned} \quad (71)$$

and thus the pair of 2-fold SUSY Hamiltonians coincides $H^+ = H^-$. They reduce to the corresponding trivial 2-fold SUSY ordinary quantum systems without reflections when $v_{0+} = v_{0-} = C_{10} = 0$.

B. The $C_{01} \neq 0$ Case

In this case, it is inevitable from (53) that

$$w_{1+} = v_{0-} = w_{0-} = 0, \quad (72)$$

and the potential terms V_0^\pm , V_1^\pm and the 2-fold supercharge component P_2^- read from (30), (31), (36), (37), and (20) as

$$4V_0^+ = 3(w_{1-})' - 2w_{0+} + (w_{1-})^2 + (v_{1-})^2 - 4C_{00}, \quad (73a)$$

$$4V_1^+ = -3(v_{1-})' - 2v_{0+} - 2w_{1-}v_{1-} - 4C_{01}, \quad (73b)$$

$$4V_0^- = -(w_{1-})' - 2w_{0+} + (w_{1-})^2 + (v_{1-})^2 - 4C_{00}, \quad (73c)$$

$$4V_1^- = (v_{1-})' - 2v_{0+} - 2w_{1-}v_{1-} - 4C_{01}, \quad (73d)$$

$$P_2^- = \frac{d^2}{dq^2} + (w_{1-} + v_{1-}\mathcal{P})\frac{d}{dq} + w_{0+} + v_{0+}\mathcal{P}. \quad (73e)$$

We note that all the potential terms $V_0^\pm(q)$ and $V_1^\pm(q)$ have even parity. The conditions (44), (46), and (A10) are automatically satisfied, and thus all the remaining conditions to be solved are (A9) and (A13) which now read as

$$2B \begin{pmatrix} w_{0+} \\ v_{0+} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_3 \end{pmatrix}, \quad B = \begin{pmatrix} 2[(w_{1-})^2 + (v_{1-})^2] & -4w_{1-}v_{1-} \\ 2w_{1-}v_{1-} & -(w_{1-})^2 - (v_{1-})^2 \end{pmatrix}, \quad (74)$$

where f_1 and f_3 are now given by

$$f_1 = -2w_{1-}(w_{1-})'' + ((w_{1-})')^2 - 2v_{1-}(v_{1-})'' + ((v_{1-})')^2 + 2(w_{1-})^2(w_{1-})' + 2(w_{1-})'(v_{1-})^2 + 4w_{1-}v_{1-}(v_{1-})' + (w_{1-})^4 + 6(w_{1-})^2(v_{1-})^2 + (v_{1-})^4 + 16C_{10}, \quad (75a)$$

$$f_3 = -(w_{1-})''v_{1-} + (w_{1-})'(v_{1-})' - w_{1-}(v_{1-})'' + 2w_{1-}(w_{1-})'v_{1-} + (w_{1-})^2(v_{1-})' + (v_{1-})^2(v_{1-})' + 2(w_{1-})^3v_{1-} + 2w_{1-}(v_{1-})^3 - 8C_{11}. \quad (75b)$$

They are simultaneous linear equations for w_{0+} and v_{0+} . Hence, we must treat the problem according to the value of $\det B = -2[(w_{1-})^2 - (v_{1-})^2]^2$.

Case 4. $(v_{1-})^2 \neq (w_{1-})^2$:

In the non-degenerate case $(v_{1-})^2 \neq (w_{1-})^2$, they are uniquely solved as

$$\begin{aligned} -2(\det B)w_{0+} &= [(w_{1-})^2 + (v_{1-})^2] f_1 - 4w_{1-}v_{1-}f_3, \\ -(\det B)v_{0+} &= w_{1-}v_{1-}f_1 - [(w_{1-})^2 + (v_{1-})^2] f_3. \end{aligned} \quad (76)$$

Hence, the most general 2-fold SUSY quantum systems with reflections in this case are expressed in terms of the two arbitrary functions having definite parity w_{1-} , and v_{1-} by the substitution of (76) for w_{0+} and v_{0+} into (73). Due to the assumption $C_{10} \neq 0$, the systems do not admit a reduction to ordinary quantum systems without reflections. When we put $C_{01} = 0$, the systems in this case reduces to the ones in Case 1 with the constraint (72).

Case 5. $v_{1-} = \pm w_{1-} \neq 0$:

In the degenerate case $v_{1-} = \pm w_{1-} \neq 0$, on the other hand, the two equations in (75) are not linearly independent and are equivalent to the following single equation:

$$4(w_{1-})^2(w_{0+} \mp v_{0+}) = -2w_{1-}(w_{1-})'' + ((w_{1-})')^2 + 4(w_{1-})^2(w_{1-})' + 4(w_{1-})^4 + 8C_{10}, \quad (77)$$

with $C_{11} = \mp C_{10}$. Hence, we can again express the most general 2-fold SUSY quantum systems with reflections in terms of two functions having definite parity, e.g., w_{1-} and w_{0+} , by eliminating the other two functions. Due to the assumption $C_{10} \neq 0$, the systems do not admit a reduction to ordinary quantum systems without reflections. When we put $C_{01} = 0$, the systems in this case reduces to the ones in Case 2 with the constraint (72).

It is worth noting that the 2-fold SUSY quantum systems with reflections characterized by the constraint (72) are essentially equivalent to the 2×2 Hermitian matrix 2-fold SUSY quantum systems in Ref. [15]. To see the relation, we first split the linear function space \mathfrak{F} in which the systems have been considered into its even and odd parts, denoted by \mathfrak{F}_+ and \mathfrak{F}_- , respectively. Their elements $\psi_+(q) \in \mathfrak{F}_+$ and $\psi_-(q) \in \mathfrak{F}_-$ are even and odd functions,

respectively, and thus $\mathcal{P} \cdot \psi_{\pm}(q) = \pm \psi_{\pm}(q)$. With this grading, we can introduce a two-component representation of $\psi \in \mathfrak{F}$ as [1]

$$\psi(q) \stackrel{\text{rep.}}{=} \begin{pmatrix} \psi_+(q) \\ \psi_-(q) \end{pmatrix}. \quad (78)$$

Then, a reflection operator \mathcal{P} , a differential operator d/dq , and any multiplicative operator of even and odd functions, $f_+(q)$ and $f_-(q)$, are represented by 2×2 matrices as

$$\begin{aligned} \mathcal{P} &\stackrel{\text{rep.}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3, & \frac{d}{dq} &\stackrel{\text{rep.}}{=} \begin{pmatrix} 0 & d/dq \\ d/dq & 0 \end{pmatrix} = \sigma_1 \frac{d}{dq}, \\ f_+ &\stackrel{\text{rep.}}{=} \begin{pmatrix} f_+ & 0 \\ 0 & f_+ \end{pmatrix} = f_+ I_2, & f_- &\stackrel{\text{rep.}}{=} \begin{pmatrix} 0 & f_- \\ f_- & 0 \end{pmatrix} = f_- \sigma_1, \end{aligned} \quad (79)$$

where σ_1 and σ_3 are the Pauli matrices and I_2 is the 2×2 unit matrix. In this representation, the 2-fold SUSY systems H^{\pm} and P_2^- in the present case (73) are expressed as

$$\begin{aligned} H^+ &\stackrel{\text{rep.}}{=} -\frac{1}{2} I_2 \frac{d^2}{dq^2} + \frac{1}{4} [3(w_{1-})' - 2w_{0+} + (w_{1-})^2 + (v_{1-})^2 - 4C_{00}] I_2 \\ &\quad - \frac{1}{4} [3(v_{1-})' + 2v_{0+} + 2w_{1-}v_{1-} - 4C_{01}] \sigma_3, \end{aligned} \quad (80a)$$

$$\begin{aligned} H^- &\stackrel{\text{rep.}}{=} -\frac{1}{2} I_2 \frac{d^2}{dq^2} + \frac{1}{4} [-(w_{1-})' - 2w_{0+} + (w_{1-})^2 + (v_{1-})^2 - 4C_{00}] I_2 \\ &\quad - \frac{1}{4} [-(v_{1-})' + 2v_{0+} + 2w_{1-}v_{1-} - 4C_{01}] \sigma_3, \end{aligned} \quad (80b)$$

$$P_2^- \stackrel{\text{rep.}}{=} I_2 \frac{d^2}{dq^2} + (w_{1-} I_2 - v_{1-} \sigma_3) \frac{d}{dq} + w_{0+} I_2 + v_{0+} \sigma_3. \quad (80c)$$

Comparing the above with the most general 2×2 Hermitian matrix 2-fold SUSY systems in Ref. [15], we find that the above 2×2 matrix 2-fold SUSY systems (80) are identical to the latter with the following substitutions:

$$\begin{aligned} w_{10} &\rightarrow w_{1-}, & v_1 &\rightarrow v_{1-}, & w_{00} &\rightarrow w_{0+}, & v_0 &\rightarrow -v_{0+}, \\ C_0 &\rightarrow C_{00}, & C_{01}, C_{02} &\rightarrow 0, & C_{03} &\rightarrow -1, & \tilde{C} &\rightarrow -C_{11}. \end{aligned} \quad (81)$$

The two conditions (75) are also identical with the corresponding ones for the latter systems (cf., Eqs. (35) and (36) in Ref. [15]) with the above substitutions. However, there exist differences between them, that is, in our present systems all the functions w_{1-} , v_{1-} , w_{0+} , and v_{0+} have definite parity but are in general complex while in the Hermitian models all the functions w_{10} , v_1 , w_{00} , and v_0 do not have definite parity in general but are restricted to be real.

V. DISCUSSION AND SUMMARY

In this article, we have for the first time formulated generically \mathcal{N} -fold SUSY in quantum mechanical systems with reflections and constructed the most general 1- and 2-fold SUSY systems. We have found in particular that there are seven inequivalent cases of 2-fold SUSY

realized by quantum systems with reflections. In Cases 1, 2, 3-3 they are characterized by three arbitrary functions having definite parity while in Cases 3-1, 3-2, 4, 5 they are by two. Furthermore, the systems in Cases 1, 3-1, and 3-3 reduce to the corresponding general 2-fold SUSY ordinary quantum systems without reflections while Cases 2, 3-2, 4, and 5 do not. Hence, it turns out that \mathcal{N} -fold SUSY in quantum systems with reflections has much richer structure than in ordinary systems without reflections. In addition to the detailed studies for $\mathcal{N} > 2$ cases, there are many future issues to be followed after this work as the followings:

1. The fact that the most general 2-fold SUSY quantum systems with reflections include as particular cases those which are essentially equivalent with 2×2 Hermitian matrix 2-fold SUSY quantum models indicates that the former could have relation to more general 2×2 non-Hermitian or higher-dimensional matrix 2-fold SUSY quantum models. Or there might exist a unified framework of \mathcal{N} -fold SUSY which includes both quantum mechanical systems with reflections and matrix models as special cases. In this respect, it is interesting to generalize the formulation of \mathcal{N} -fold SUSY to quantum mechanical matrix models with reflection operators.

2. It is important to clarify general aspects of \mathcal{N} -fold SUSY in quantum systems with reflections, as were done in [4, 5] for ones without reflections. In the latter case, there are two significant features, namely, the equivalence between \mathcal{N} -fold SUSY and weak quasi-solvability and the equivalence between the conditions (9) and (10). In the case of 2-fold SUSY quantum systems with reflections investigated in Section IV, however, the latter equivalence would be violated since the condition (42) coming from (9) is evidently weaker than the conditions (43) and (44) coming from (10). That was exactly the reason why we considered the both to derive (45) and (46). In this respect, it is interesting to study what happens when we employ only one of the conditions (9) and (10) exclusively. It is also worth applying the general approach for quantum systems without reflections recently proposed by us in Ref. [19] to ones with reflections.

3. In the case without reflections, the systematic algorithm for constructing an \mathcal{N} -fold SUSY system [10] based on quasi-solvability has shown to be quite effective. Hence, its generalization to the present case with reflections is desirable. It would enable us to connect directly the possible types of such \mathcal{N} -fold SUSY systems with the possible linear spaces of functions preserved by a second-order linear differential operator with reflections. It would also help us to clarify the structure of the little -1 Jacobi polynomials and to obtain the family of polynomial systems which arise as a set of exact eigenfunctions of an operator of this kind. To the best of our knowledge, there have been no systematic investigations into quasi-solvable operators involving reflection operators. We would report some results on this subject in our subsequent publications.

4. Shape invariance is a well-known sufficient condition for *solvability* of one-dimensional Schrödinger equations [20]. It means in particular that it always implies \mathcal{N} -fold SUSY. In fact, some shape-invariant potentials in the case without reflections were systematically constructed as particular cases of \mathcal{N} -fold SUSY with intermediate Hamiltonians [21, 22]. To the best of knowledge, there have been no investigations into shape-invariant potentials with reflections, and we expect that our formulation of \mathcal{N} -fold SUSY would be also quite efficient in constructing systematically shape-invariant quantum systems with reflections.

5. Extension to more general second-order linear differential operators with reflections would be possible. In particular, a quantum mechanical model with reflections having position-dependent mass would be an interesting candidate as a natural generalization of \mathcal{N} -fold SUSY in ordinary quantum systems with position-dependent mass formulated in Ref. [23].

6. In the case without reflections, there are several intimate relations between \mathcal{N} -fold SUSY and \mathcal{N} th-order paraSUSY [21, 22, 24, 25]. We expect that we can formulate higher-order paraSUSY in quantum systems with reflections in a way such that the relations to \mathcal{N} -fold SUSY in the case without reflections remain intact in the latter case. Extension of higher-order \mathcal{N} -fold paraSUSY [26] to quantum systems with reflections would be also possible.

Appendix A: List of Formulas

The components of anti-commutators of 2-fold supercharges:

$$\begin{aligned}
P_2^- P_2^+ &= \frac{d^4}{dq^4} + 2v_{1+} \mathcal{P} \frac{d^3}{dq^3} + [-3w_1' + 2w_0 - (w_1)^2 - (v_1)^2 + (-3(v_{1\mathcal{P}})' + 2v_{0+} \\
&\quad - w_1 v_{1\mathcal{P}} - w_{1\mathcal{P}} v_1) \mathcal{P}] \frac{d^2}{dq^2} + [-3w_1'' + 2w_0' - 2w_1 w_1' - 2v_1 v_1' + (3(v_{1\mathcal{P}})'' - 2(v_{0\mathcal{P}})' \\
&\quad + 2(w_{1\mathcal{P}})' v_1 + 2w_1 (v_{1\mathcal{P}})' - w_1 v_{0\mathcal{P}} - w_{1\mathcal{P}} v_0 + w_0 v_{1\mathcal{P}} + w_{0\mathcal{P}} v_1) \mathcal{P}] \frac{d}{dq} - w_1''' + w_0'' - w_1 w_1'' \\
&\quad - v_1 v_1'' - w_1' w_0 + w_1 w_0' + v_1' v_0 - v_1 v_0' + (w_0)^2 + (v_0)^2 + (-(v_{1\mathcal{P}})''' + (v_{0\mathcal{P}})'' - (w_{1\mathcal{P}})'' v_1 \\
&\quad - w_1 (v_{1\mathcal{P}})'' + (w_{1\mathcal{P}})' v_0 + w_1 (v_{0\mathcal{P}})' - (w_{0\mathcal{P}})' v_1 - w_0 (v_{1\mathcal{P}})' + w_0 v_{0\mathcal{P}} + w_{0\mathcal{P}} v_0) \mathcal{P}, \tag{A1}
\end{aligned}$$

$$\begin{aligned}
P_2^+ P_2^- &= \frac{d^4}{dq^4} + 2v_{1+} \mathcal{P} \frac{d^3}{dq^3} + [w_1' + 2w_0 - (w_1)^2 - (v_{1\mathcal{P}})^2 + (-2v_1' - (v_{1\mathcal{P}})' + 2v_{0+} \\
&\quad + w_1 v_1 + w_{1\mathcal{P}} v_{1\mathcal{P}}) \mathcal{P}] \frac{d^2}{dq^2} + [w_1'' + 2w_0' - 2w_1 w_1' - 2v_{1\mathcal{P}} (v_{1\mathcal{P}})' + (v_1'' - 2v_0' - w_1' v_1 \\
&\quad - (w_{1\mathcal{P}})' v_{1\mathcal{P}} - w_1 v_1' - w_{1\mathcal{P}} (v_{1\mathcal{P}})' + w_1 v_0 + w_{1\mathcal{P}} v_{0\mathcal{P}} + w_0 v_1 + w_{0\mathcal{P}} v_{1\mathcal{P}}) \mathcal{P}] \frac{d}{dq} + w_0'' \\
&\quad - w_1' w_0 - w_1 w_0' - (v_{1\mathcal{P}})' v_{0\mathcal{P}} - v_{1\mathcal{P}} (v_{0\mathcal{P}})' + (w_0)^2 + (v_{0\mathcal{P}})^2 + (v_0'' - w_1' v_0 - w_1 v_0' \\
&\quad - (w_{0\mathcal{P}})' v_{1\mathcal{P}} - w_{0\mathcal{P}} (v_{1\mathcal{P}})' + w_0 v_0 + w_{0\mathcal{P}} v_{0\mathcal{P}}) \mathcal{P}. \tag{A2}
\end{aligned}$$

The three conditions obtained from the substitution of (30), (31), (36), and (37) into (27), (34), and (40):

$$\begin{aligned} & w_1''' - w_1 w_1'' - 2(w_1')^2 + 4w_1' w_0 + 2w_1 w_0' - v_{1-}(v_{1-})'' - 2((v_{1-})')^2 \\ & - 2(v_{1-})'(2v_{0+} - v_{0-}) - 2v_{1-}(v_{0+})' + 4v_{0+}v_{0-} - 2(w_1)^2 w_1' - 2(w_{1-})'(v_{1-})^2 \\ & - 2(w_{1+} + 2w_{1-})v_{1-}(v_{1-})' + 4w_{1-}v_{1-}v_{0-} + 8C_{01}v_{0-} = 0, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} & 2w_1''' - 5(w_1')^2 + 8w_1' w_0 + 4w_1 w_0' - 5((v_{1-})')^2 - 4(v_{1-})'(2v_{0+} - v_{0-}) \\ & - 4v_{1-}v_0' + 4(2v_{0+} + v_{0-})v_{0-} - 6(w_1)^2 w_1' + 4(w_1)^2 w_0 - 6w_1'(v_{1-})^2 \\ & - 12w_{1-}v_{1-}(v_{1-})' - 8w_{1-}v_{1-}v_{0+} + 4w_0(v_{1-})^2 - (w_1)^4 \\ & - 2[(w_1)^2 + 2(w_{1-})^2](v_{1-})^2 - (v_{1-})^4 - 16C_{10} = 0, \end{aligned} \quad (\text{A4})$$

$$\begin{aligned} & 2w_1''' - 4w_1 w_1'' - 3(w_1')^2 + 8w_1' w_0 + 4w_1 w_0' - 4v_{1-}(v_{1-})'' - 3((v_{1-})')^2 \\ & - 4(v_{1-})'(2v_{0+} - v_{0-}) - 4v_{1-}(v_{0\mathcal{P}})' + 4v_{0-}(2v_{0+} - v_{0-}) - 2(w_1)^2 w_1' \\ & - 4(w_1)^2 w_0 - 2w_1'(v_{1-})^2 - 4w_{1-}v_{1-}(v_{1-})' + 8w_{1-}v_{1-}v_{0+} - 4w_0(v_{1-})^2 \\ & + (w_1)^4 + 2[(w_1)^2 + 2(w_{1-})^2](v_{1-})^2 + (v_{1-})^4 + 16C_{10} = 0. \end{aligned} \quad (\text{A5})$$

The three conditions obtained from the substitution of (30), (31), (36), and (37) into (28), (35), and (41):

$$\begin{aligned} & (v_{1-})''' + 2(v_{0-})'' - w_1'' v_{1-} - 4(w_{1-})'(v_{1-})' + w_{1\mathcal{P}}(v_{1-})'' - 2(w_{1+} + 2w_{1-})'v_0 \\ & - 2w_1(v_{0+})' + 2(w_{0\mathcal{P}})'v_{1-} + 2(2w_{0+} - w_{0-})(v_{1-})' + 4w_{0-}v_{0-} \\ & - 2[2w_{1-}(w_{1-})' + w_{1\mathcal{P}}(w_{1+})']v_{1-} - 2w_1 w_{1-}(v_{1-})' - 4w_{1+}w_{1-}v_0 \\ & - 4w_{1-}w_{0-}v_{1-} - 2(v_{1-})^2(v_{1-})' - 8C_{01}w_{0-} = 0, \end{aligned} \quad (\text{A6})$$

$$\begin{aligned} & (v_{1-})''' + 2(v_{0-})'' + 2(w_{1+})''v_{1-} - 5(w_{1-})'(v_{1-})' - 2w_{1+}(v_{1-})'' - 2(w_{1+})'v_0 \\ & - 2(w_{1-})'(2v_{0+} - v_{0-}) - 2w_1(v_{0\mathcal{P}})' + 2(w_{0\mathcal{P}})'v_{1-} + 2(2w_{0+} - w_{0-})(v_{1-})' \\ & + 4w_{0-}v_{0-} - 6w_{1-}(w_{1-})'v_{1-} - 3[(w_{1+})^2 + (w_{1-})^2](v_{1-})' - 2[(w_{1+})^2 + (w_{1-})^2]v_{0+} \\ & + 4w_{1-}w_{0+}v_{1-} - 3(v_{1-})^2(v_{1-})' - 2(v_{1-})^2v_{0+} - 2[(w_{1+})^2 + (w_{1-})^2]w_{1-}v_{1-} \\ & - 2w_{1-}(v_{1-})^3 + 8C_{11} = 0, \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} & (v_{1-})''' + 2(v_{0-})'' - 2(w_{1-})''v_{1-} - 3(w_{1-})'(v_{1-})' - 2w_{1-}(v_{1-})'' - 2w_1'v_0 \\ & - (2w_{1-})'v_{0+} - 2w_1v_0' + 2(w_{0\mathcal{P}})'v_{1-} + 2(2w_{0+} - w_{0-})(v_{1-})' + 4w_{0-}v_{0-} \\ & - 2w_{1-}(w_{1-})'v_{1-} - [(w_{1+})^2 + (w_{1-})^2](v_{1-})' + 2[(w_{1+})^2 + (w_{1-})^2]v_{0+} \\ & - 4w_{1-}w_{0+}v_{1-} - (v_{1-})^2(v_{1-})' + 2(v_{1-})^2v_{0+} + 2[(w_{1+})^2 + (w_{1-})^2]w_{1-}v_{1-} \\ & + 2w_{1-}(v_{1-})^3 - 8C_{11} = 0. \end{aligned} \quad (\text{A8})$$

The even and odd parts of the condition (47):

$$\begin{aligned}
& 2w_{1+}(w_{1+})'' + 2w_{1-}(w_{1-})'' - ((w_{1+})')^2 - ((w_{1-})')^2 + 2v_{1-}(v_{1-})'' \\
& - ((v_{1-})')^2 + 4(v_{0-})^2 - 2[(w_{1+})^2 + (w_{1-})^2](w_{1-})' - 4w_{1+}w_{1-}(w_{1+})' \\
& + 4[(w_{1+})^2 + (w_{1-})^2]w_{0+} + 8w_{1+}w_{1-}w_{0-} - 2(w_{1-})'(v_{1-})^2 - 4w_{1-}v_{1-}(v_{1-})' \\
& - 8w_{1-}v_{1-}v_{0+} + 4w_{0+}(v_{1-})^2 - (w_{1+})^4 - 6(w_{1+})^2(w_{1-})^2 - (w_{1-})^4 \\
& - 2[(w_{1+})^2 + 3(w_{1-})^2](v_{1-})^2 - (v_{1-})^4 - 16C_{10} = 0, \tag{A9}
\end{aligned}$$

$$\begin{aligned}
& (w_{1+})''w_{1-} - (w_{1+})'(w_{1-})' + w_{1+}(w_{1-})'' - 2w_{1+}w_{1-}(w_{1-})' - [(w_{1+})^2 \\
& + (w_{1-})^2](w_{1+})' + 2[(w_{1+})^2 + (w_{1-})^2]w_{0-} + 4w_{1+}w_{1-}w_{0+} - w_{1+}v_{1-}(v_{1-})' \\
& - 2w_{1+}v_{1-}v_{0+} - 2(w_{1+})^3w_{1-} - 2w_{1+}(w_{1-})^3 - 2w_{1+}w_{1-}(v_{1-})^2 = 0, \tag{A10}
\end{aligned}$$

where (44) and (46) have been used to eliminate $(v_{0-})'$ in (A10).

The even and odd parts of the condition (48):

$$\begin{aligned}
& (w_{1-})''' - w_{1+}(w_{1+})'' - w_{1-}(w_{1-})'' - 2((w_{1+})')^2 - 2((w_{1-})')^2 + 4(w_{1+})'w_{0-} \\
& + 2w_{1+}(w_{0-})' + 4(w_{1-})'w_{0+} + 2w_{1-}(w_{0+})' - v_{1-}(v_{1-})'' - 2((v_{1-})')^2 \\
& - 4(v_{1-})'v_{0+} - 2v_{1-}(v_{0+})' - 2[(w_{1+})^2 + (w_{1-})^2](w_{1-})' - 4w_{1+}w_{1-}(w_{1+})' \\
& - 2(w_{1-})'(v_{1-})^2 - 4w_{1-}v_{1-}(v_{1-})' = 0, \tag{A11}
\end{aligned}$$

$$\begin{aligned}
& (w_{1+})''' - (w_{1+})''w_{1-} - 4(w_{1+})'(w_{1-})' - w_{1+}(w_{1-})'' + 4(w_{1+})'w_{0+} \\
& + 4(w_{1-})'w_{0-} + 2w_{1+}(w_{0+})' + 2w_{1-}(w_{0-})' + 2(v_{1-})'v_{0-} + 4v_{0+}v_{0-} \\
& - 2[(w_{1+})^2 + (w_{1-})^2](w_{1+})' - 4w_{1+}w_{1-}(w_{1-})' - 2(w_{1+})'(v_{1-})^2 = 0. \tag{A12}
\end{aligned}$$

The even and odd parts of the condition (50):

$$\begin{aligned}
& (w_{1-})''v_{1-} - (w_{1-})'(v_{1-})' + w_{1-}(v_{1-})'' - [w_{1+}(w_{1+})' + 2w_{1-}(w_{1-})']v_{1-} \\
& - (w_{1-})^2(v_{1-})' - 2(w_{1-})^2v_{0+} + 2w_{1+}w_{0-}v_{1-} + 4w_{1-}w_{0+}v_{1-} - (v_{1-})^2(v_{1-})' \\
& - 2(v_{1-})^2v_{0+} - 2[(w_{1+})^2 + (w_{1-})^2]w_{1-}v_{1-} - 2w_{1-}(v_{1-})^3 + 8C_{11} = 0, \tag{A13}
\end{aligned}$$

$$(w_{1+})''v_{1-} - w_{1+}(v_{1-})'' + 2(w_{1-})'v_{0-} + 2w_{1-}(v_{0-})' = 0, \tag{A14}$$

where (44) and (46) have been used to eliminate $(v_{0-})'$ in (A13).

The even and odd parts of the condition (51):

$$\begin{aligned}
& (v_{1-})''' - (w_{1-})''v_{1-} - 4(w_{1-})'(v_{1-})' - w_{1-}(v_{1-})'' - 2(w_{1+})'v_{0-} \\
& - 4(w_{1-})'v_{0+} - 2w_{1-}(v_{0+})' + 2(w_{0+})'v_{1-} + 4w_{0+}(v_{1-})' + 4w_{0-}v_{0-} \\
& - 4w_{1-}(w_{1-})'v_{1-} - 2[(w_{1+})^2 + (w_{1-})^2](v_{1-})' - 2(v_{1-})^2(v_{1-})' = 0, \tag{A15}
\end{aligned}$$

$$\begin{aligned}
& 2(v_{0-})'' + (w_{1+})''v_{1-} - w_{1+}(v_{1-})'' - 2(w_{1+})'v_{0+} - 2w_{1+}(v_{0+})' \\
& - 2(w_{0-})'v_{1-} - 2w_{0-}(v_{1-})' = 0. \tag{A16}
\end{aligned}$$

The definition of f_i ($i = 1, 2, 3$):

$$\begin{aligned} f_1 = & -2w_{1+}(w_{1+})'' - 2w_{1-}(w_{1-})'' + ((w_{1+})')^2 + ((w_{1-})')^2 - 2v_{1-}(v_{1-})'' \\ & + ((v_{1-})')^2 - 4(v_{1-})^2 + 2[(w_{1+})^2 + (w_{1-})^2](w_{1-})' + 4w_{1+}w_{1-}(w_{1+})' \\ & + 2(w_{1-})'(v_{1-})^2 + 4w_{1-}v_{1-}(v_{1-})' + (w_{1+})^4 + 6(w_{1+})^2(w_{1-})^2 + (w_{1-})^4 \\ & + 2[(w_{1+})^2 + 3(w_{1-})^2](v_{1-})^2 + (v_{1-})^4 + 16C_{10}, \end{aligned} \quad (A17)$$

$$\begin{aligned} f_2 = & -(w_{1+})''w_{1-} + (w_{1+})'(w_{1-})' - w_{1+}(w_{1-})'' + 2w_{1+}w_{1-}(w_{1-})' \\ & + [(w_{1+})^2 + (w_{1-})^2](w_{1+})' + w_{1+}v_{1-}(v_{1-})' + 2(w_{1+})^3w_{1-} \\ & + 2w_{1+}(w_{1-})^3 + 2w_{1+}w_{1-}(v_{1-})^2, \end{aligned} \quad (A18)$$

$$\begin{aligned} f_3 = & -(w_{1-})''v_{1-} + (w_{1-})'(v_{1-})' - w_{1-}(v_{1-})'' + [w_{1+}(w_{1+})' \\ & + 2w_{1-}(w_{1-})']v_{1-} + (w_{1-})^2(v_{1-})' + (v_{1-})^2(v_{1-})' + 2[(w_{1+})^2 \\ & + (w_{1-})^2]w_{1-}v_{1-} + 2w_{1-}(v_{1-})^3 - 8C_{11}. \end{aligned} \quad (A19)$$

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- [1] S. Post, L. Vinet, and A. Zhedanov, J. Phys. A: Math. Theor. 44 (2011) 435301. arXiv:1107.5844 [math-ph].
 - [2] L. Vinet and A. Zhedanov, J. Phys. A: Math. Theor. 44 (2011) 085201. arXiv:1011.1669 [math.CA].
 - [3] A. A. Andrianov, M. V. Ioffe, and V. P. Spiridonov, Phys. Lett. A 174 (1993) 273. arXiv:hep-th/9303005.
 - [4] H. Aoyama, M. Sato, and T. Tanaka, Nucl. Phys. B 619 (2001) 105. arXiv:quant-ph/0106037.
 - [5] A. A. Andrianov and A. V. Sokolov, Nucl. Phys. B 660 (2003) 25. arXiv:hep-th/0301062.
 - [6] T. Tanaka, In Morris B. Levy, ed., Mathematical Physics Research Developments (Nova Science Publishers, Inc., New York, 2009), chapter 18. pp. 621–679.
 - [7] H. Aoyama, M. Sato, and T. Tanaka, Phys. Lett. B 503 (2001) 423. arXiv:quant-ph/0012065.
 - [8] T. Tanaka, Nucl. Phys. B 662 (2003) 413. arXiv:hep-th/0212276.
 - [9] A. González-López and T. Tanaka, Phys. Lett. B 586 (2004) 117. arXiv:hep-th/0307094.
 - [10] A. González-López and T. Tanaka, J. Phys. A: Math. Gen. 38 (2005) 5133. arXiv:hep-th/0405079.
 - [11] T. Tanaka, J. Math. Phys. 51 (2010) 032101. arXiv:0910.0328 [math-ph].
 - [12] S. Samuel and J. Wess, Nucl. Phys. B 221 (1983) 153.
 - [13] D. V. Volkov and V. P. Akulov, JETP Lett. 16 (1972) 438.
 - [14] I. Antoniadis, E. Dudas, D. M. Ghilencea, and P. Tziveloglou, Nucl. Phys. B 841 (2010) 157. arXiv:1006.1662 [hep-th].
 - [15] T. Tanaka, Mod. Phys. Lett. A 27 (2012) 1250051. arXiv:1108.0480 [math-ph].
 - [16] A. G. Nikitin and Y. Karadzhov, J. Phys. A: Math. Theor. 44 (2011) 305204. arXiv:1101.4129 [math-ph].
 - [17] A. A. Andrianov, M. V. Ioffe, F. Cannata, and J. P. Dedonder, Int. J. Mod. Phys. A 10 (1995) 2683. arXiv:hep-th/9404061.
 - [18] A. A. Andrianov, M. V. Ioffe, and D. N. Nishnianidze, Phys. Lett. A 201 (1995) 103. arXiv:hep-th/9404120.
 - [19] T. Tanaka, J. Phys. A: Math. Theor. 44 (2011) 465301. arXiv:1107.1035 [math-ph].

- [20] L. É. Gendenshtein, JETP Lett. 38 (1983) 356.
- [21] B. Bagchi and T. Tanaka, Ann. Phys. 324 (2009) 2438. arXiv:0905.4330 [hep-th].
- [22] B. Bagchi and T. Tanaka, Ann. Phys. 325 (2010) 1679. arXiv:1002.1766 [hep-th].
- [23] T. Tanaka, J. Phys. A: Math. Gen. 39 (2006) 219. arXiv:quant-ph/0509132.
- [24] T. Tanaka, Ann. Phys. 322 (2007) 2350. arXiv:hep-th/0610311.
- [25] T. Tanaka, Ann. Phys. 322 (2007) 2682. arXiv:hep-th/0612263.
- [26] T. Tanaka, Mod. Phys. Lett. A 22 (2007) 2191. arXiv:hep-th/0611008.